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A shallow water equation as a geodesic flow on the Bott–Virasoro group

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Abstract

The Camassa–Holm equation is shown to give rise to a geodesic flow of a certain right invariant metric on the Bott–Virasoro group. The sectional curvature of this metric is computed and shown to assume positive and negative signs.

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1. Introduction

In [OK] Ovsienko and Khesin showed that the solutions to the periodic Korteweg–de Vries equation can be interpreted as geodesics of the right invariant metric on the Bott–Virasoro group which at the identity is given by the L^2 inner product. Below we show that an analogous correspondence can be established for another completely integrable nonlinear partial differential equation recently introduced by Camassa and Holm [CH]

$$\partial_t u + 2\kappa\partial_x u - \partial_x^2\partial_t u + 3u\partial_x u - 2\partial_x u\partial_x^2 u - u\partial_x^3 u = 0, \tag{1.1}$$

where κ is a constant. Camassa and Holm derived this equation indirectly using an asymptotic expansion in the Hamiltonian of the Euler equations of hydrodynamics. The equation describes the motion of shallow water waves. The global well-posedness of the initial value problem for (1.1) is not yet fully understood, although it is known that for some initial

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conditions its solutions develop singularities in finite time (see [CH] or [CHH]). The interpretation given below of (1.1) as a geodesic equation on the Bott–Virasoro group may bring in some additional insight into the existence problem for this equation.

2. Metric and its geodesic flow

We proceed to describe the necessary background. Let $\mathcal{D}^s(S^1)$ be the group of orientation preserving diffeomorphisms of the circle which are of Sobolev class H^s and let $\mathrm{Vect}^s(S^1) = T_e\mathcal{D}^s(S^1)$ be the corresponding Lie algebra of H^s vector fields on S^1. Throughout we assume s to be sufficiently large so that $\mathcal{D}^s(S^1)$ can be equipped with a structure of an infinite-dimensional manifold and our formal computations can be rigorously justified.

Recall that the Bott–Virasoro group $\widehat{\mathcal{D}}^s(S^1)$ is the non-trivial central extension of $\mathcal{D}^s(S^1)$ with the group operation given by

$$\widehat{\eta} \circ \widehat{\xi} = \left(\eta \circ \xi, \alpha + \beta + \int_{S^1} \log \partial_x(\eta \circ \xi)\, \mathrm{d} \log \partial_x \xi \right), \tag{2.1}$$

where $\widehat{\eta} = (\eta, \alpha)$, $\widehat{\xi} = (\xi, \beta)$ with $\eta, \xi \in \mathcal{D}^s(S^1)$ and $\alpha, \beta \in \mathrm{R}$ and where the term given by the integral is a 2-cocycle on $\mathcal{D}^s(S^1)$ computed by Bott [B].

Further, recall that the corresponding Virasoro algebra, $\widehat{\mathrm{Vect}}^s(S^1)$, is the non-trivial central extension of $\mathrm{Vect}^s(S^1)$. The general 2-cocycle, determining this extension, was found by Gelfand and Fuchs [GF] (see also [PS, Section 4.2]). The commutator in the Virasoro algebra is given by

$$[\widehat{V}, \widehat{W}] = -\left((v\partial_x w - w\partial_x v)\frac{\partial}{\partial x}, \int_{S^1} v(\partial_x^3 w + \partial_x w)\, \mathrm{d}x \right), \tag{2.2}$$

where

$$\widehat{V} = \left(v\frac{\partial}{\partial x}, a \right), \quad \widehat{W} = \left(w\frac{\partial}{\partial x}, b \right) \quad \text{with } a, b \in \mathrm{R} \text{ and } v\frac{\partial}{\partial x}, w\frac{\partial}{\partial x} \in T_e\mathcal{D}^s(S^1).$$

The minus sign in the formula above makes it the correct choice for a bracket in the Lie algebra of right invariant vector fields on the group.

On the Virasoro algebra consider the H^1 inner product

$$\langle \widehat{V}, \widehat{W} \rangle_{H^1} = \int_{S^1} \partial_x v \partial_x w\, \mathrm{d}x + \int_{S^1} vw\, \mathrm{d}x + ab, \tag{2.3}$$

where $\widehat{V}$ and $\widehat{W}$ are as above. Extend (2.3) to a right invariant metric on the Bott–Virasoro group by setting

$$\langle \widehat{V}, \widehat{W} \rangle_{\widehat{\xi}} = \langle \mathrm{d}_{\widehat{\xi}} R_{\widehat{\xi}^{-1}} \widehat{V}, \mathrm{d}_{\widehat{\xi}} R_{\widehat{\xi}^{-1}} \widehat{W} \rangle_{H^1} \tag{2.4}$$

for any $\widehat{\xi} \in \widehat{\mathcal{D}^s}(S^1)$ and $\widehat{V}, \widehat{W} \in T_{\widehat{\xi}}\widehat{\mathcal{D}^s}(S^1)$ and where $R_{\widehat{\xi}} : \widehat{\mathcal{D}^s}(S^1) \to \widehat{\mathcal{D}^s}(S^1)$ is the right translation by $\widehat{\xi}$, $R_{\widehat{\xi}}(\widehat{\eta}) = \widehat{\eta} \circ \widehat{\xi}$. An explicit formula for the derivative, $\mathrm{d}_{\widehat{\xi}} R_{\widehat{\xi}^{-1}}$, can be obtained from (2.1) (cf. [M2]).

Our main result is as follows.

Theorem 1. *Let $t \to \widehat{\eta}(t)$ be a curve in the Bott–Virasoro group starting at $\widehat{\eta}(0) = (e, 0)$ in the direction $\dot{\widehat{\eta}}(0) = (v_0 \frac{\partial}{\partial x}, a_0)$. Then $\widehat{\eta}$ is a geodesic of the H^1 metric (2.4) if and only if the pair $(v(t)\frac{\partial}{\partial x}, a(t)) = \mathrm{d}_{\widehat{\eta}} R_{\widehat{\eta}^{-1}} \dot{\widehat{\eta}}(t)$ satisfies*

$$\partial_t a = 0, \qquad \partial_t v - \partial_x^2 \partial_t v = v\partial_x^3 v + 2\partial_x v \partial_x^2 v - 3v\partial_x v - a\partial_x^3 v - a\partial_x v \tag{2.5}$$

with $a(0) = a_0$ and $v(0) = v_0$.

Remark 1. It is readily seen that, for a suitable choice of a constant c, the substitution $v \to v + c$ transforms the second of the equations in (2.5) into the Camassa–Holm equation (1.1).

Proof of Theorem 1. First recall the following fundamental result about geodesic flows on arbitrary Lie groups.

Proposition 1. *Let G be a (possibly infinite-dimensional) Lie group equipped with a metric $\langle \cdot, \cdot \rangle$ which is invariant under right translations $R_g : G \to G$, $R_g(h) = h \cdot g$. A curve $t \to \gamma(t)$ in G is a geodesic of this metric if and only if $u(t) = \mathrm{d}_{\gamma_t} R_{\gamma_t^{-1}} \dot{\gamma}(t)$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = -ad^*_{u(t)} u(t), \tag{2.6}$$

where ad^*_u is the adjoint of ad_u with respect to the metric $\langle \cdot, \cdot \rangle$, that is for any u, v and $w \in T_e G$

$$\langle ad^*_u v, w \rangle_e = \langle v, [u, w] \rangle_e.$$

The proof of Proposition 1 can be found in [A, Appendix 2] or [MR, Section 13.8].

Remark 2. Eq. (2.6) is sometimes called the Euler–Poincaré–Arnold equation. In its equivalent formulation on the dual $T_e^* G$ it is known as the Lie–Poisson equation.

We shall show that Eq. (2.5) is precisely the Euler–Poincare–Arnold equation on the Virasoro algebra $\widehat{\mathrm{Vect}}^s(S^1)$ associated with the H^1 inner product (2.3). Given any

$$\widehat{V} = \left(v\frac{\partial}{\partial x}, a\right), \quad \widehat{W} = \left(w\frac{\partial}{\partial x}, b\right), \quad \widehat{U} = \left(u\frac{\partial}{\partial x}, c\right) \qquad \text{in } \widehat{\mathrm{Vect}}^s(S^1),$$

one computes from (2.2) and (2.3)

$$\begin{aligned}\langle ad^*_{\widehat{V}}\widehat{W},\widehat{U}\rangle_{H^1} &= \langle \widehat{W},[\widehat{V},\widehat{U}]\rangle_{H^1}\\ &= -\int_{S^1}\partial_x w\partial_x(v\partial_x u - u\partial_x v)\,\mathrm{d}x - \int_{S^1} w(v\partial_x u - u\partial_x v)\,\mathrm{d}x\\ &\quad -\int_{S^1} bv(\partial_x^3 u + \partial_x u)\,\mathrm{d}x.\end{aligned}$$

Integrating by parts and using the fact that the functions v, w, u are periodic, this expression can be written as

$$\begin{aligned}&\int_{S^1} u(-v\partial_x^3 w - 2\partial_x v\partial_x^2 w + v\partial_x w + 2w\partial_x v + b\partial_x^3 v + b\partial_x v)\,\mathrm{d}x\\ &\quad = \int_{S^1} u(1-\partial_x^2)(1-\partial_x^2)^{-1}\\ &\quad\quad \times \{-v\partial_x^3 w - 2\partial_x v\partial_x^2 w + v\partial_x w + 2w\partial_x v + b\partial_x^3 v + b\partial_x v\}\,\mathrm{d}x,\end{aligned}$$

from which we obtain the formula for the coadjoint operator

$$\begin{aligned}ad^*_{\widehat{V}}\widehat{W} = \Big((1-\partial_x^2)^{-1}\\ \times \{-v\partial_x^3 w - 2\partial_x v\partial_x^2 w + v\partial_x w + 2w\partial_x v + b\partial_x^3 v + b\partial_x v\}\frac{\partial}{\partial x}, 0\Big).\end{aligned} \tag{2.7}$$

Let $\widehat{\eta}(t)$ be the geodesic described in the statement of Theorem 1. Using (2.6) and (2.7) we now obtain the corresponding Euler–Poincare–Arnold equation

$$\partial_t a = 0, \qquad (1-\partial_x^2)\partial_t v = v\partial_x^3 v + 2\partial_x v\partial_x^2 v - 3v\partial_x v - a\partial_x^3 v - a\partial_x v,$$

and Theorem 1 follows from Proposition 1. □

3. Sectional curvature

Without much extra effort it is possible to write down an expression for the sectional curvature of the metric (2.4). Observe that since by construction right translations preserve the metric it is sufficient to do the computation at the identity

$$\widehat{e} = (e, 0) \in \widehat{\mathcal{D}^s}(S^1).$$

Theorem 2. *The sectional curvature at the identity in an arbitrary two plane detrmined by a pair*

$$\widehat{V} = \left(v\frac{\partial}{\partial x}, a\right) \quad \textit{and} \quad \widehat{W} = \left(w\frac{\partial}{\partial x}, b\right)$$

is given by

$$
\begin{aligned}
&\langle R(\widehat{V}, \widehat{W})\widehat{W}, \widehat{V}\rangle_{H^1} \\
&= \int_{S^1} \{\tfrac{1}{4}(v\partial_x^3 w + w\partial_x^3 v + 2\partial_x v\partial_x^2 w + 2\partial_x w\partial_x^2 v - 3v\partial_x w - 3w\partial_x v \\
&\quad - b\partial_x^3 v - a\partial_x^3 w - b\partial_x v - a\partial_x w)(1-\partial_x^2)^{-1}(v\partial_x^3 w + w\partial_x^3 v + 2\partial_x v\partial_x^2 w \\
&\quad + 2\partial_x w\partial_x^2 v - 3v\partial_x w - 3w\partial_x v - b\partial_x^3 v - a\partial_x^3 w - b\partial_x v - a\partial_x w) \\
&\quad - (v\partial_x^3 v + 2\partial_x v\partial_x^2 v - 3v\partial_x v - a\partial_x^3 v - a\partial_x v)(1-\partial_x^2)^{-1}(w\partial_x^3 w \\
&\quad + 2\partial_x w\partial_x^2 w - 3w\partial_x w - b\partial_x^3 w - b\partial_x w) - \tfrac{1}{4}(v\partial_x^2 w - w\partial_x^2 v)^2 \\
&\quad - \tfrac{5}{4}(v\partial_x w - w\partial_x v)^2 - \tfrac{1}{2}(v\partial_x w - w\partial_x v)(\partial_x v\partial_x^2 w - \partial_x w\partial_x^2 v \\
&\quad - b\partial_x^3 v + a\partial_x^3 w - b\partial_x v + a\partial_x w)\}\, \mathrm{d}x \\
&\quad - \frac{3}{4}\left(\int_{S^1} v(\partial_x^3 w + \partial_x w)\, \mathrm{d}x\right)^2 .
\end{aligned}
\tag{3.1}
$$

Proof. This is just an extended exercise in integration by parts using (2.7), periodicity and the standard formula for the curvature at the identity of a left invariant (and therefore right invariant) metric on a Lie group

$$
\begin{aligned}
&\langle R(\widehat{V}, \widehat{W})\widehat{W}, \widehat{V}\rangle_{H^1} \\
&= \tfrac{1}{4}\|ad^*_{\widehat{V}}\widehat{W} + ad^*_{\widehat{W}}\widehat{V}\|^2_{H^1} - \langle ad^*_{\widehat{V}}\widehat{V}, ad^*_{\widehat{W}}\widehat{W}\rangle_{H^1} - \tfrac{3}{4}\|[\widehat{V}, \widehat{W}]\|^2_{H^1} \\
&\quad - \tfrac{1}{2}\langle[[\widehat{V}, \widehat{W}], \widehat{W}], \widehat{V}\rangle_{H^1} - \tfrac{1}{2}\langle[[\widehat{W}, \widehat{V}], \widehat{V}], \widehat{W}\rangle_{H^1}
\end{aligned}
$$

(see e.g. [CE, Chapter 3]). □

Note that the curvatures in (3.1) can take on both signs. As expected, the sectional curvature in the plane containing an element from the centre of the algebra is non-negative and a straightforward inspection shows that for $\widehat{V} = (0, a)$ and $\widehat{W} = (\cos 2x\frac{\partial}{\partial x}, 0)$ it is in fact positive. On the other hand we have:

Corollary 3. *Let*

$$
\widehat{V} = \left(\sin kx\frac{\partial}{\partial x}, 0\right) \quad \textit{and} \quad \widehat{W} = \left(\cos kx\frac{\partial}{\partial x}, 0\right),
$$

then

$$
\begin{aligned}
&\langle R(\widehat{V}, \widehat{W})\widehat{W}, \widehat{V}\rangle_{H^1} \\
&= \frac{\pi k^2}{4(1+4k^2)}(8 - 3\pi - (8+6\pi)k^2 + (2+21\pi)k^4 - 12\pi k^6).
\end{aligned}
$$

Therefore if $k \geq 2$ the sectional curvature is strictly negative.

Remark 3. The formula given in Theorem 2, though a bit lengthy, allows one to study directly the behaviour of geodesics of the metric (2.4) on the Bott–Virasoro group and therefore indirectly, by Theorem 1, solutions of the Camassa–Holm equation. It would be of interest for example to investigate its possible relevance for the stability problem of the initial value problem for (1.1) as in the case of the Euler equations of hydrodynamics (cf. [A] or [M1]).

Remark 4. From the computations presented above it follows that the particular case ($\kappa = 0$) of Eq. (1.1), also studied in [CHH], can be interpreted as a geodesic flow on just $\mathcal{D}(S^1)$ of the metric which at the identity is H^1.

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